Measures in wavelet decompositions

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Abstract

In applications, choices of orthonormal bases in Hilbert space $\mathcal{H}$ may come about from the simultaneous diagonalization of some specific abelian algebra of operators. This is the approach of quantum theory as suggested by John von Neumann; but as it turns out, much more recent constructions of bases in wavelet theory, and in dynamical systems, also fit into this scheme. However, in these modern applications, the basis typically comes first, and the abelian algebra might not even be made explicit. It was noticed recently that there is a certain finite set of non-commuting operators F_i, first introduced by engineers in signal processing, which helps to clarify this connection, and at the same time throws light on decomposition possibilities for wavelet packets used in pyramid algorithms. There are three interrelated components to this: an orthonormal basis, an abelian algebra, and a projection-valued measure. While the operators F_i were originally intended for quadrature mirror filters of signals, recent papers have shown that they are ubiquitous in a variety of modern wavelet constructions, and in particular in the selection of wavelet packets from libraries of bases. These are constructions which make a selection of a basis with the best frequency concentration in signal or data-compression problems. While the algebra $\mathcal{A}$ generated by the F_i-system is non-abelian, and goes under the name "Cuntz algebra" in C^*-algebra theory, each of its representations contains a canonical maximal abelian subalgebra, i.e., the subalgebra is some $C(X)$ for a Gelfand space X. A given representation of $\mathcal{A}$, restricted to $C(X)$, naturally induces a projection-valued measure on X, and each vector in $\mathcal{H}$ induces a scalar-valued measure on X. We develop this construction in the general context with a view to wavelet applications, and we show that the measures that had been studied earlier for a very restrictive class of F_i-systems (i.e., the Lemarié-Meyer quadrature mirror filters) in the theory of wavelet packets are special cases of this. Moreover, we prove a structure theorem for certain classes of induced scalar measures. In the applications, X may be the unit interval, or a Cantor set; or it may be an affine

fractal, or even one of the more general iteration limits involving iterated function systems consisting of conformal maps.

Key words: Hilbert space, Cuntz algebra, completely positive map, creation operators, wavelet packets, pyramid algorithm, product measures, orthogonality relations, equivalence of measures, iterated function systems (IFS), scaling function, multiresolution, subdivision scheme, singular measures, absolutely continuous measures
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1 Introduction

A popular approach to wavelet constructions is based on a so-called scaling identity, or scaling equation. A solution to this equation is a function on $\mathbb{R}^d$ for some d. The equation is related to a subdivision scheme that is used in numerical analysis and in computer graphics. In that language, it arises from a fixed scaling matrix, assumed expansive, a system of masking coefficients, and a certain subdivision algorithm. An iteration of the scaling produces a succession of subdivisions into smaller and smaller frequency bands. In signal processing, the coefficients in the equation refer to "frequency response". There are various refinements, however, of this setup: two such refinements are *multi-wavelets* and *singular systems*.

If the masking coefficients are turned into a generating function, called a low-pass filter m_0, then the scaling identity takes a form which admits solutions with an infinite product representation. Various regularity assumptions are usually placed on the function m_0. The first requirement is usually that the solution, i.e., the scaling function, is in $L^2(\mathbb{R}^d)$, but other Hilbert spaces of functions on $\mathbb{R}^d$ are also considered. If the number of masking coefficients is finite, then m_0 is a Fourier polynomial. (For the Daubechies wavelet, there are four coefficients, and $d = 1$.) Readers not familiar with wavelets are referred to the classic [1] by Daubechies. More general families of multiresolutions are studied in [2], [3], and [4]. For recent applications of multiresolutions to physics, see [5]. In general, however, m_0 might be a fairly singular function. In favorable cases, the associated infinite product will be the Fourier transform of the scaling function. This function, sometimes called the father function, is the starting point of most wavelet constructions, the multiresolution schemes.

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The function m_0 is a function of one or more frequency variables, and convergence of the associated infinite product dictates requirements on m_0 for small frequencies, hence low-pass. The term "low-pass" suggests a filter which lets low-frequency signals pass with high probability. A complete system, of which m_0 is a part, and which is built from appropriately selected frequency bands, offers an effective tool for wavelet analysis and for signal processing. Such a system gives rise to operators F_i, and their duals F_i^*, that are the starting point for a class of algorithms called pyramid algorithms. They are basic to both signal processing and the analysis of wavelet packets. (In operator theory, F_i^* is usually denoted S_i, and S_i^* is set equal to F_i. The reason is that it is the operator F_i^* that is isometric.) In the more traditional approaches, m_0 is a Fourier polynomial, or at least a Lipschitz-class function on a suitable torus, and the low-pass signal analysis is then relatively well understood. But a variety of applications, for example to multi-wavelets, dictate filters m_0 that are no better than continuous, or perhaps only measurable. Then the standard tools break down, and probabilistic and operator theoretic methods are forced on us. This is the setting which is the focus of the present paper.

Recent developments in wavelet analysis have brought together ideas from engineering and from computational mathematics, as well as fundamentals from representation theory. One of the aims of this paper is to stress the interconnections, as opposed to one aspect of this in isolation.

By now, the subject draws on ideas from a variety of directions. Of these directions, we single out quadrature-mirror filters from signal/image processing, see Figure 1.1 below. High-pass/low-pass signal-processing algorithms have now been adopted by pure mathematicians, although they historically first were intended for speech signals, see [7]. Perhaps unexpectedly, essentially the same quadrature relations were rediscovered in operator algebra theory, and they are now used in relatively painless constructions of varieties of wavelet bases. The connection to signal processing is rarely stressed in the math literature. Yet, the flow of ideas between signal processing and wavelet mathematics is a success story that deserves to be told. Without these recent synergistic trends, we would perhaps only know isolated examples of wavelets. Thus, mathematicians have borrowed from engineers; and the engineers may be happy to know that what they do is used in mathematics.

Our new results in this paper include Corollary 3.11, Proposition 5.3, Theorem 6.3, and Corollary 5.5, covering both construction (algorithms) for wavelets, and selection (statistics) of the "best" wavelets in explicitly parametrized families. They concern a construction of measures which allows the selection of the "best" wavelet from a library of wavelet bases (decomposition theory).

It is well known that the quadrature mirror filters which are used in subband constructions of signal processing are also the building blocks for wavelets and

for wavelet packets; see, e.g., [6] and [7]. The reader may find good accounts of recent results on wavelet packets in the papers [4] and [3]. The scaling function for the wavelets, and the wavelet packet functions arise from pyramid algorithms which are built directly from the quadrature mirror filters. While the wavelet functions live in spaces of functions on $\mathbb{R}$, typically $L^2(\mathbb{R})$, the signals may be analyzed in the sequence space $\ell^2(\mathbb{Z})$, or equivalently $L^2(\mathbb{T})$, where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. As is well known, the isomorphism $L^2(\mathbb{T}) \cong \ell^2(\mathbb{Z})$ is given by the transform of Fourier series. Then there is an operator which maps $\ell^2(\mathbb{Z})$ onto some resolution subspace in $L^2(\mathbb{R})$ and intertwines the analysis of the signals in ℓ^2 with the transformations acting on the wavelet functions. In the simplest case, there is a function $\varphi \in L^2(\mathbb{R})$, called the scaling function, which sets up the operator from ℓ^2 to $L^2(\mathbb{R})$: If $\xi = (\xi_k)_{k\in\mathbb{Z}} \in \ell^2$, set

$$(W_\varphi \xi)(x) = \sum_{k\in\mathbb{Z}} \xi_k \varphi(x-k). \tag{1.1}$$

A subband filter is given by a sequence $(a_k)_{k\in\mathbb{Z}}$ of *frequency response coefficients*. They define an operator S_0 on ℓ^2 as follows:

$$(S_0\xi)_n = \sum_k a_{n-2k}\xi_k\,, \tag{1.2}$$

and it is denoted [filter]ⓣ, i.e., it is a composition of the two operations, with ⓣ being the symbol for up-sampling; see [8] and [9] for details. A function φ on $\mathbb{R}$ is said to satisfy a scaling identity with *masking coefficients* $(a_k)_{k\in\mathbb{Z}}$ if

$$\varphi(x) = \sqrt{2}\sum_{k\in\mathbb{Z}} a_k \varphi(2x-k). \tag{1.3}$$

The following lemma makes the connection between the discrete analysis of ℓ^2 and the wavelet analysis on $\mathbb{R}$.

The issue of smoothness properties of the possible scaling functions φ, and the corresponding wavelets, is an important one. It is studied in a number of papers, for example in [10], and the reader will find more in [9].

Lemma 1.1. *Suppose the sequence $(a_k)_{k\in\mathbb{Z}}$ is such that the function*

$$m_0(z) = \sum_k a_k z^k \tag{1.4}$$

is in $L^\infty(\mathbb{T})$. Let S_0 be the corresponding bounded operator on ℓ^2. Let $\varphi \in L^2(\mathbb{R})$, and let W_φ be the corresponding operator (1.1). *Then φ satisfies the scaling identity* (1.3) *if and only if*

$$W_\varphi S_0 \xi = \frac{1}{\sqrt{2}}(W_\varphi\xi)\left(\frac{x}{2}\right). \tag{1.5}$$

In other words, W_φ intertwines S_0 with the dyadic scaling operator on $L^2(\mathbb{R})$. We shall introduce

$$(Uf)(x) = \frac{1}{\sqrt{2}} f\left(\frac{x}{2}\right) \tag{1.6}$$

for the unitary scaling operator on $L^2(\mathbb{R})$, and (1.5) takes the form

$$W_\varphi S_0 = U W_\varphi. \tag{1.7}$$

PROOF. The proof is straightforward, and we refer to [9] or [11] for details. □

The quadrature conditions on the filter (a_k) may be stated as

$$\sum_k \bar{a}_k a_{k+2l} = \delta_{0,l}, \qquad l \in \mathbb{Z}. \tag{1.8}$$

If

$$m_0(z) = \sum_k a_k z^k \quad \text{and} \quad m_1(z) = z\,\overline{m_0(-z)}, \qquad z \in \mathbb{T}, \tag{1.9}$$

then the two operators S_0 and S_1 given by

$$(S_i f)(z) = m_i(z) f\left(z^2\right), \\ f \in L^2(\mathbb{T}),\ z \in \mathbb{T} = \{\, z \in \mathbb{C} \mid |z| = 1 \,\},\ i = 0,1, \tag{1.10}$$

define isometries on $L^2(\mathbb{T})$, and they satisfy the relations

$$\sum_i S_i S_i^* = \mathbb{1}_{L^2(\mathbb{T})}, \tag{1.11}$$

$$S_i^* S_j = \delta_{i,j} \mathbb{1}_{L^2(\mathbb{T})}, \tag{1.12}$$

where $\mathbb{1}_{L^2(\mathbb{T})}$ denotes the identity operator in the Hilbert space $L^2(\mathbb{T})$, and $\mathbb{T}$ is equipped with the usual Haar measure. Equivalently, $L^2(\mathbb{T})$ is viewed as a space of 2π-periodic functions, and the measure on $\mathbb{T}$ is then $(2\pi)^{-1}\,d\theta$. The relations (1.11)–(1.12) are called the Cuntz relations, see Section 3 below, but they also reflect the realization of the diagram in Figure 1.1, from signal processing.

When the two operators and their dual adjoints act on sequences, then (1.11) takes the form

$$S_0 S_0^* \xi + S_1 S_1^* \xi = \xi \tag{1.13}$$

and expresses perfect reconstruction of signals from the subbands.

In view of (1.11)–(1.12) it is clear that the isometries S_i provide dyadic subdivisions of the Hilbert space $\mathcal{H} = L^2(\mathbb{T}) \cong \ell^2$. Specifically, for every $k \in \mathbb{Z}_+$ the subspaces

$$\mathcal{H}(i_1, i_2, \ldots, i_k) := S_{i_1} S_{i_2} \cdots S_{i_k} \mathcal{H} \tag{1.14}$$

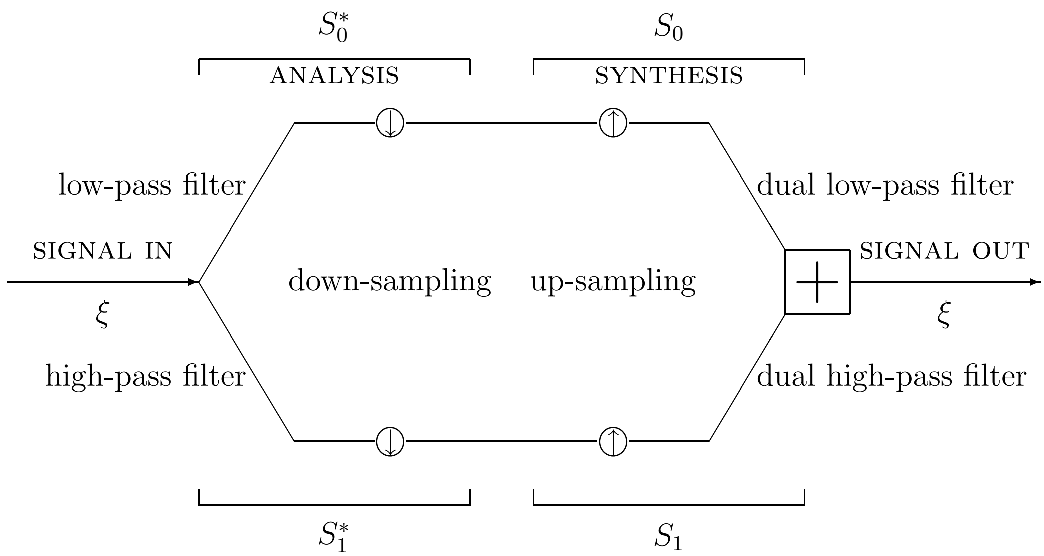

Fig. 1.1. Perfect reconstruction of signals

are mutually orthogonal, and

$$\sum_{i_1,\dots,i_k}^{\oplus} \mathcal{H}\left(i_1, i_2, \dots, i_k\right) = \mathcal{H}. \tag{1.15}$$

But if the index labels $(i_1, \dots, i_k)$ are used in assigning dyadic partitions, for example the intervals $\left[\frac{i_1}{2} + \dots + \frac{i_k}{2^k}, \frac{i_1}{2} + \dots + \frac{i_k}{2^k} + \frac{1}{2^k}\right)$, then it can be shown that $(i_1, \dots, i_k) \mapsto \mathcal{H}\left(i_1, \dots, i_k\right)$ extends to a projection-valued measure E on the unit interval I, defined on the Borel subsets of I, specifically

$$E\left(\left[\frac{i_1}{2} + \dots + \frac{i_k}{2^k}, \frac{i_1}{2} + \dots + \frac{i_k}{2^k} + \frac{1}{2^k}\right)\right) = \mathcal{H}\left(i_1, \dots, i_k\right), \tag{1.16}$$

and it was shown in [6] and [12] that this measure determines the selection of bases of wavelet packets from some prescribed library of bases. The libraries of bases in turn are determined by quadrature mirror filters.

However, it is difficult to compute $E\left(\,\cdot\,\right)$ in general. If $f \in \mathcal{H}$, $\|f\| = 1$, then

$$\mu_f\left(\,\cdot\,\right) := \left\langle\, f \mid E\left(\,\cdot\,\right) f \,\right\rangle = \left\|E\left(\,\cdot\,\right) f\right\|^2 \tag{1.17}$$

is a probability measure on I, and it is easier to compute for special classes of quadrature mirror filters; explicit results are given in [6] and [12] for the filters m_0, m_1 first introduced by Y. Meyer. But it is not known in general for which quadrature mirror filters m_i, and for which $f \in \mathcal{H}$, the measure $\mu_f\left(\,\cdot\,\right) = \left\|E\left(\,\cdot\,\right) f\right\|^2$ is absolutely continuous with respect to Lebesgue measure on I. Absolute continuity is desirable in the calculus of libraries of bases formed from wavelet packets.

Remark 1.2. While the conditions we list in (1.11)–(1.12) may seem unnecessarily stringent, it is possible to use the methods in our paper on a wider class of operator systems S_i than the ones which correspond to perfect reconstruction, as we define it by Figure 1.1. In fact, Arveson [13] has recently developed an elegant operator-theoretic approach to finite systems of operators S_i, $i = 0, 1, \ldots, n$, when it is only assumed that the operator system of $n+1$ operators forms a *row-contraction*. By this we mean that each operator S_i is defined in a Hilbert space $\mathcal{H}$, and the system satisfies the contractivity condition

$$\left\| \sum_{i=0}^{n} S_i f_i \right\|^2 \leq \sum_{i=0}^{n} \|f_i\|^2 \qquad \text{for all } (f_0, \ldots, f_n) \in \bigoplus_0^n \mathcal{H},$$

or equivalently

$$\sum_{i=0}^{n} \|S_i^* f\|^2 \leq \|f\|^2 \qquad \text{for all } f \in \mathcal{H}.$$

As stressed in papers by Ron and Shen, e.g., [14], such row-contractions arise from conditions on systems of filter functions which are weaker than the ones we summarize in equations (1.8)–(1.9) above. The corresponding function system in $L^2(\mathbb{R})$ will then not be a wavelet system in the sense we discuss below. It will only have considerably weaker orthogonality properties than those which are customary for the standard wavelet bases, and the authors of [14] refer to these systems as *framelets*; see also our survey paper [15].

2 Subdivisions

Subdivisions serve as an effective tool in the theory of dynamical systems [16], in computations [17], and in approximation theory [14]; see also [9]. Moreover, they are part of many wavelet constructions: see, e.g., [18]. The simplest such is the familiar representations of the fractions $0 \leq x < 1$ in base 2. For $k \in \mathbb{N}$ and $a_1, a_2, \ldots, a_k \in \{0, 1\}$, set

$$J_k(a) := \left[\frac{a_1}{2} + \cdots + \frac{a_k}{2^k}, \frac{a_1}{2} + \cdots + \frac{a_k}{2^k} + \frac{1}{2^k} \right). \tag{2.1}$$

Each interval $J_k(a)$ is contained in some $J_{k-1}(b)$, and the length of $J_k(a)$ is 2^{-k} by definition. Moreover, the symbols $(a_1, \ldots, a_k)$, for k finite, yield a one-to-one representation of the dyadic rational fractions. Note that we are excluding those infinite strings which terminate with an infinite tail of 1's, and an infinite tail of 0's may be omitted in listing the bits $a_1, a_2, \ldots, a_k$.

We will also need the analogous representation of fractions in base N where $N \in \mathbb{Z}_+$, $N \geq 2$. In that case $a_i \in \{0, 1, \ldots, N-1\}$, the left-hand endpoint of

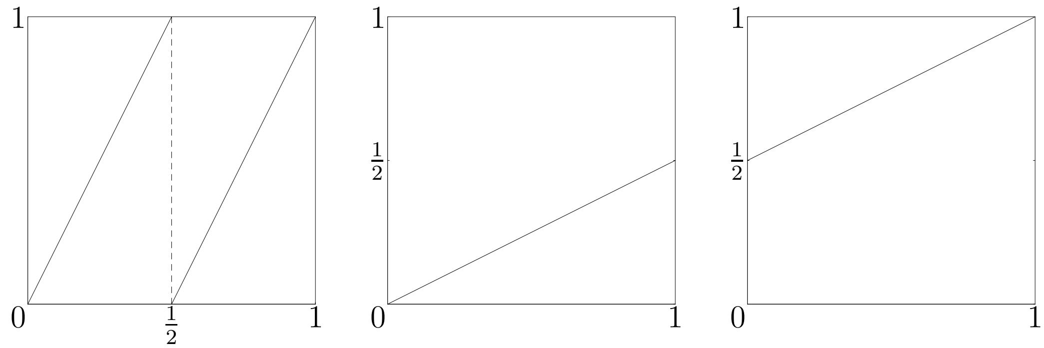

Fig. 2.1. Graphic representation of σ, σ_0, and σ_1.

$J_k(a)$ is $\dfrac{a_1}{N} + \cdots + \dfrac{a_k}{N^k}$, and length $(J_k(a)) = N^{-k}$.

More general partitions like this arise in the study of endomorphisms $\sigma\colon X \to X$, where X is a compact Hausdorff space, and σ is continuous and onto. If, for each $x \in X$, the cardinality of $\sigma^{-1}(x) = \{\, y \in X \mid \sigma(y) = x \,\}$ is N, independently of x, then there are branches of the inverse, i.e., maps

$$\sigma_0, \sigma_1, \ldots, \sigma_{N-1}\colon X \longrightarrow X \tag{2.2}$$

such that

$$\sigma \circ \sigma_i = 1_X, \tag{2.3}$$

or in other notation,

$$\sigma(\sigma_i(x)) = x, \qquad x \in X, \tag{2.4}$$

for $0 \le i < N$. Naturally, it is of special interest if the sections $\{\sigma_i\}_{0 \le i < N}$ may be chosen to be continuous, as is the case in the study of complex iteration of rational maps; see, e.g., [19], [20].

Example 2.1. The particular example, $N = 2$, mentioned above arises this way when the identification

$$[0,1) \cong \mathbb{R}/\mathbb{Z} \tag{2.5}$$

is made and $\sigma(x) = 2x \bmod 1$. The three maps σ, σ_0, and σ_1 may then be represented by the graphs in Figure 2.1.

If $J = [0,1)$ is the usual unit interval on the line, then the subdivision from (2.1) takes the form

$$J_k(a) = \sigma_{a_1} \circ \sigma_{a_2} \circ \cdots \circ \sigma_{a_k}(J), \tag{2.6}$$

and the system

$$\sigma_a := \sigma_{a_1} \circ \sigma_{a_2} \circ \cdots \circ \sigma_{a_k} \tag{2.7}$$

forms a set of branches for $\sigma^k = \underbrace{\sigma \circ \cdots \circ \sigma}_{k \text{ times}}$ as $a_i \in \{0,1\}$, $i = 1, \ldots, k$.

Example 2.2. Set $\sigma_0(x) = x/3$ and $\sigma_1(x) = (x+2)/3$, and let $X \subset \mathbb{R}$ be the unique solution to

$$X = \sigma_0(X) \cup \sigma_1(X). \tag{2.8}$$

Then X is the familiar middle-third Cantor set, and there is a unique Borel probability measure μ supported on X and satisfying

$$\mu = \frac{1}{2}\left(\mu \circ \sigma_0^{-1} + \mu \circ \sigma_1^{-1}\right) \tag{2.9}$$

or equivalently

$$\int f\, d\mu = \frac{1}{2}\left(\int f \circ \sigma_0\, d\mu + \int f \circ \sigma_1\, d\mu\right) \qquad \text{for all } f \in C(X). \tag{2.10}$$

3 Projection-valued measures

In this section we study subdivisions of compact metric spaces, and subdivisions of projections in Hilbert space, and we use our observations in the construction of certain projection-valued measures.

Definition 3.1. Let (X, d) be a compact metric space. For subsets $A \subset X$, we define the *diameter*

$$|A| := \sup\{\, d(x,y) \mid x, y \in A \,\}. \tag{3.1}$$

A *partition* of X is a family $\{A(i)\}_{i \in I}$, I some index set, such that

$$\bigcup_i A(i) = X, \quad \text{and} \quad A(i) \cap A(j) = \varnothing \text{ if } i \neq j. \tag{3.2}$$

Let $N \in \mathbb{Z}_+$, $N \geq 2$. Let $\Gamma_N := \{0, 1, \dots, N-1\}$. Suppose for each $k \in \mathbb{Z}_+$, we have a partition into Borel subsets $\{A_k(a)\}$ indexed by $a \in \Gamma_N^k = \underbrace{\Gamma_N \times \cdots \times \Gamma_N}_{k \text{ times}}$, and

$$|A_k(a)| = O\left(N^{-ck}\right), \qquad c > 0. \tag{3.3}$$

Suppose every $A_{k+1}(a)$ is contained in some $A_k(b)$. We then say that $\{A_k(a)\}$ is an *N-adic system of partitions* of X.

Definition 3.2. Let $\mathcal{H}$ be a complex Hilbert space. A *partition of projections* in $\mathcal{H}$ is a system $\{P(i)\}_{i \in I}$ of projections, i.e., $P(i) = P(i)^* = P(i)^2$, such that

$$P(i)\,P(j) = 0 \text{ if } i \neq j \quad \text{and} \quad \sum_i P(i) = \mathbb{1}_{\mathcal{H}}, \tag{3.4}$$

where $\mathbb{1}_{\mathcal{H}}$ denotes the identity operator in $\mathcal{H}$. Let $N \in \mathbb{Z}_+$, $N \geq 2$. Suppose for each $k \in \mathbb{Z}_+$ we have a partition of projections $\{P_k(a)\}_{a \in \Gamma_N^k}$ such that every $P_{k+1}(a)$ is contained in some $P_k(b)$, i.e.,

$$P_k(b)\, P_{k+1}(a) = P_{k+1}(a)\,. \tag{3.5}$$

Then the combined system $\{P_k(a)\}_{k \in \mathbb{Z}_+,\ a \in \Gamma_N^k}$ is a system of partitions of $\mathbb{1}_{\mathcal{H}}$ forming, by N-adic subdivisions, an N-adic system of projections. We refer to this system as an *N-adic system of partitions of $\mathbb{1}_{\mathcal{H}}$ into projections* (or for brevity, if the context makes clear what Hilbert space $\mathcal{H}$ is being partitioned into projections, simply as an *N-adic system of projections*).

Definition 3.3. We use $\mathcal{B}(X)$ to denote the Borel subsets of the compact metric space X. A positive operator-valued function E defined on $\mathcal{B}(X)$ may be called a *σ-additive measure* if, given a sequence $B_1, B_2, \dots$ in $\mathcal{B}(X)$ such that $B_i \cap B_j = \varnothing$ for $i \neq j$, the measures combine according to the formula

$$E\left(\bigcup_i B_i\right) = \sum_i E(B_i)\,, \tag{3.6}$$

where, since the values $E(B_i)$ are positive operators, we may take the summation on the right-hand side to be convergent in the strong operator topology. Such a measure is called an *orthogonal projection-valued measure* if it satisfies the additional properties (i)–(iii):

(i) $E(B) = E(B)^* = E(B)^2$ for $B \in \mathcal{B}(X) =$ the Borel subsets of X,
(ii) $E(B_1)\, E(B_2) = 0$ if $B_1, B_2 \in \mathcal{B}(X)$ satisfy $B_1 \cap B_2 = \varnothing$, and
(iii) $E(X) = \mathbb{1}_{\mathcal{H}}$.

Remark 3.4. There are four independent conditions in Definition 3.3. If $E(\,\cdot\,)$ is a function defined on $\mathcal{B}(X)$ and taking values in positive operators on $\mathcal{H}$, and if only property (3.6) is satisfied, we say $E(\,\cdot\,)$ is a positive operator-valued measure. If (i) is *also* satisfied, we say that E is projection-valued. If (3.6), (i), and (ii) are satisfied, we say that the projection-valued measure is orthogonal. If all four conditions hold, we talk of a projection-valued measure which is orthogonal and normalized, in short an *orthogonal projection-valued measure*. In this paper, we will only have occasion to study the case when $E(\,\cdot\,)$ satisfies all four conditions.

Lemma 3.5. *Let $N \in \mathbb{Z}_+$, $N \geq 2$. Let (X, d) be a compact metric space, and let $\mathcal{H}$ be a complex Hilbert space. Let $\{A_k(a)\}_{k \in \mathbb{Z}_+,\, a \in \Gamma_N^k}$ be an N-adic system of partitions of X, and let $\{P_k(a)\}_{k \in \mathbb{Z}_+,\, a \in \Gamma_N^k}$ be the corresponding N-adic system of partitions of $\mathbb{1}_{\mathcal{H}}$ into projections (i.e., the corresponding N-adic system of projections). Then there is a unique normalized orthogonal projection-valued measure $E(\,\cdot\,)$ defined on the Borel subsets of X and taking values in the*

orthogonal projections in $\mathcal{H}$ such that

$$E\left(A_k\left(a\right)\right) = P_k\left(a\right) \qquad \text{for all } k \in \mathbb{Z}_+ \text{ and } a \in \Gamma_N^k. \tag{3.7}$$

Definition 3.6. Let $N \in \mathbb{Z}_+$, $N \geq 2$. We shall need the *Cuntz algebra* $\mathcal{O}_N$ [21] on N generators $s_0, s_1, \dots, s_{N-1}$. It is the unique C^*-algebra on the relations

$$\sum_i s_i s_i^* = \mathbb{1} \tag{3.8}$$

where $\mathbb{1}$ is the unit element in the algebra $\mathcal{O}_N$. We have

$$s_i^* s_j = \delta_{i,j}\mathbb{1}. \tag{3.9}$$

To specify a *representation of* $\mathcal{O}_N$ on a Hilbert space $\mathcal{H}$ we need N isometries $S_0, S_1, \dots, S_{N-1}$ such that

$$\sum_i S_i S_i^* = \mathbb{1}_{\mathcal{H}}. \tag{3.10}$$

Then $S_i^* S_j = \delta_{i,j}\mathbb{1}_{\mathcal{H}}$ and the representation is determined uniquely.

Lemma 3.7. *Let $N \in \mathbb{Z}_+$, $N \geq 2$, and let $S_0, S_1, \dots, S_{N-1}$ be a representation of $\mathcal{O}_N$ on a Hilbert space $\mathcal{H}$. For $k \in \mathbb{Z}_+$ and $a = (a_1, \dots, a_k) \in \Gamma_N^k$, set*

$$S_a := S_{a_1} \cdots S_{a_k} \quad \text{and} \quad P_k\left(a\right) = S_a S_a^*. \tag{3.11}$$

Then the combined system $\{P_k\left(a\right)\}_{k\in\mathbb{Z}_+,\, a\in\Gamma_N^k}$ is an N-adic system of partitions of $\mathbb{1}_{\mathcal{H}}$ into projections (i.e., an N-adic system of projections).

We now turn to the proof of the two lemmas.

PROOF OF LEMMA 3.5. Let $N \in \mathbb{Z}_+$, and let systems $\{A_k\left(a\right)\}_{k\in\mathbb{Z}_+,\, a\in\Gamma_N^k}$ and $\{P_k\left(a\right)\}_{k\in\mathbb{Z}_+,\, a\in\Gamma_N^k}$ be given as in the statement of the lemma. For every $k \in \mathbb{Z}_+$, the finite sums

$$\sum_{a\in\Gamma_N^k} C_a \chi_{A_k(a)} \tag{3.12}$$

form an algebra $\mathfrak{A}_k$ of functions on X, and from the definition of the partition system $\{A_k\left(a\right)\}_{k\in\mathbb{Z}_+,\, a\in\Gamma_N^k}$ it follows that there are natural embeddings $\mathfrak{A}_k \hookrightarrow \mathfrak{A}_{k+1}$. From the definition of the projection system $\{P_k\left(a\right)\}_{k\in\mathbb{Z}_+,\, a\in\Gamma_N^k}$ we conclude that the mappings, defined for each $k \in \mathbb{Z}_+$,

$$\sum_{a\in\Gamma_N^k} C_a \chi_{A_k(a)} \longmapsto \sum_{a\in\Gamma_N^k} C_a P_k\left(a\right) \tag{3.13}$$

extends to the algebra

$$\mathfrak{A} := \bigcup_{k\in\mathbb{Z}_+} \mathfrak{A}_k. \tag{3.14}$$

But the operators on the right-hand side in (3.13) form an abelian algebra $\mathcal{C}$ of operators. The algebra $\mathfrak{A}$ is closed under complex conjugation $f \mapsto \bar{f}$, and $\mathcal{C}$ is $*$-closed, i.e., $E \in \mathcal{C} \Rightarrow E^* \in \mathcal{C}$. Let the mapping obtained from (3.13) be denoted π. Then one checks from the two definitions, Definition 3.1 and Definition 3.2, that $\pi(f_1 f_2) = \pi(f_1)\pi(f_2)$, $f_1, f_2 \in \mathfrak{A}$, and $\pi(\bar{f}) = \pi(f)^*$, $f \in \mathfrak{A}$. Since the sets $A_k(a)$ satisfy $|A_k(a)| = O\left(N^{-k}\right)$, where $|\cdot|$ denotes the diameter, it is clear that every $f \in C(X)$ may be approximated uniformly with a sequence in $\mathfrak{A}$. This means that $f \mapsto \pi(f)$ extends uniquely $\tilde{\pi}\colon C(X) \to B(\mathcal{H})$, and the extension satisfies

$$\tilde{\pi}(f_1 f_2) = \tilde{\pi}(f_1)\tilde{\pi}(f_2), \; f_1, f_2 \in C(X), \quad \text{and} \quad \tilde{\pi}(\bar{f}) = \tilde{\pi}(f)^*. \tag{3.15}$$

A standard argument from function theory now shows that $\tilde{\pi}$ extends further from $C(X)$ to all the Baire functions, and the extension satisfies the same multiplication rules (3.15). For this part of the argument see, e.g., [22, Section 6]. If $B \in \mathcal{B}(X)$, we may then define a projection-valued measure $E(\cdot)$ as follows:

$$E(B) := \tilde{\pi}(\chi_B), \tag{3.16}$$

where χ_B denotes the indicator function of the set B. Since $\tilde{\pi}$ is obtained as a unique extension from (3.13) it follows immediately that $E(\cdot)$ in (3.16) has the properties (i)–(iii) from Definition 3.3, and that it is countably additive, see (3.6). Moreover, it satisfies (3.7), and is determined uniquely by (3.7). □

PROOF OF LEMMA 3.7. The details are essentially well known; see, e.g., [23]. In fact, an inspection shows that the projections $P_k(a) = S_a S_a^*$, $a \in \Gamma_N^k$, introduced in (3.11) generate an abelian algebra of operators. It is a special case of the algebra $\mathcal{C}$ which we introduced in the proof of Lemma 3.5 above. Also note the following monotonicity: If S and T are positive operators on $\mathcal{H}$, we say that $S \leq T$ if

$$\langle x \mid Sx \rangle \leq \langle x \mid Tx \rangle \quad \text{holds for all } x \in \mathcal{H}. \tag{3.17}$$

The inner product of $\mathcal{H}$ is denoted $\langle \cdot \mid \cdot \rangle$ and is assumed linear in the second factor. Using the defining relation (3.10) for the generators of a representation of $\mathcal{O}_N$, note that if $a \in \Gamma_N^k$ for some k, and if $i \in \Gamma_N$, then

$$(ai) \in \Gamma_N^{k+1} \quad \text{and} \quad \sum_i P_{k+1}(ai) = P_k(a). \tag{3.18}$$

As a result, we get $P_{k+1}(ai) \leq P_k(a)$, or equivalently $P_k(a) P_{k+1}(ai) = P_{k+1}(ai)$, which is the desired relation (3.5) from Definition 3.2. □

Remark 3.8. If $\{S_i\}_{i \in \Gamma_N}$ is a representation of $\mathcal{O}_N$ on a Hilbert space, it is known that the C^*-algebra $\mathcal{C}$ generated by the projections $P_k(a) = S_a S_a^*$,

$a \in \Gamma_N^k$, $k \in \mathbb{Z}_+$, is naturally isomorphic to the algebra of all continuous functions on the infinite Cartesian

$$\text{product} \quad \prod_{\mathbb{Z}_+} \Gamma_N \quad \text{or} \quad \Gamma_N^{\mathbb{Z}_+} \tag{3.19}$$

when the Cartesian product is equipped with the product topology of Tychonoff, i.e., $\mathcal{C} \cong C(X)$ with $X = \Gamma_N^{\mathbb{Z}_+}$. Recall that X is compact; see, e.g., [24].

It is easy to see that $X = \Gamma_N^{\mathbb{Z}_+}$ carries a system of functions $\sigma, \sigma_0, \dots, \sigma_{N-1}$ with the properties listed in (2.3). If elements x in X are represented as sequences $(x_1, x_2, \dots)$, we set

$$\sigma(x_1, x_2, \dots) = (x_2, x_3, \dots) \quad \text{and} \quad \sigma_i(x_1, x_2, \dots) = (i, x_1, x_2, \dots). \tag{3.20}$$

All $N+1$ functions are continuous $X \to X$ and satisfy (2.3), i.e.,

$$\sigma(\sigma_i(x)) = x \qquad \text{for all } x \in X \text{ and } i = 0, 1, \dots, N-1. \tag{3.21}$$

There is a family of measures on X which generalizes the property (2.9) above. They are the product measures: Let $\{p_i\}_{i \in \Gamma_N}$ be given such that $p_i \geq 0$ and $\sum_i p_i = 1$, and let $\xi_k \colon X \to \Gamma_N$ be the coordinate projection $\xi_k(x_1, x_2, \dots) = x_k$. For subsets $T \subset \Gamma_N$, set

$$\xi_k^{-1}(T) = \{\, x \in X \mid \xi_k(x) \in T \,\}. \tag{3.22}$$

Using standard measure theory [24], note that there is a unique measure μ_p on X such that

$$\mu_p\left(\xi_k^{-1}(\{i\})\right) = p_i \qquad \text{for all } k \in \mathbb{Z}_+,\ i \in \Gamma_N. \tag{3.23}$$

Introducing the maps $\sigma_i \colon X \to X$ of (3.20) we note that μ_p satisfies

$$\mu_p = \sum_{i \in \Gamma_N} p_i \mu_p \circ \sigma_i^{-1}, \tag{3.24}$$

or equivalently

$$\int_X f \, d\mu_p = \sum_{i \in \Gamma_N} p_i \int_X f \circ \sigma_i \, d\mu_p \qquad \text{for all } f \in C(X). \tag{3.25}$$

Finally, note that distinct probabilities (p_i) and (p_i') yield measures μ_p and $\mu_{p'}$ which are mutually singular.

If $(S_i)_{0 \leq i < N}$ is a representation of $\mathcal{O}_N$ for some $N \in \mathbb{Z}_+$, $N \geq 2$, then we will denote the corresponding projection-valued measure on $\Gamma_N^{\mathbb{Z}_+}$ by E. If

$\{A_k(a)\}_{k\in\mathbb{Z}_+,\,a\in\Gamma_N^k}$ is an N-adic system of partitions of some compact metric space (X,d), then the corresponding projection-valued measure on $\mathcal{B}(X)$ will be denoted $E^A(\,\cdot\,)$ to stress its dependence on the partition system.

The next lemma shows that the algebra $\mathcal{C}$ in the proof of Lemma 3.5 is isomorphic to $C\left(\Gamma_N^{\mathbb{Z}_+}\right)$ where $\mathcal{C}$ is viewed as a C^*-algebra, and the infinite Cartesian product $X=\Gamma_N^{\mathbb{Z}_+}$ is given its Tychonoff topology. Since $\mathcal{C}$ is an abelian C^*-algebra we know that it is isomorphic to $C(K)$ for some compact Hausdorff space K. The isomorphism $\mathcal{C}\cong C(K)$ is called the Gelfand transform, and K the Gelfand space. The conclusion of the lemma is that $\Gamma_N^{\mathbb{Z}_+}$ is the Gelfand space of $\mathcal{C}$, and further we offer a formula for the Gelfand transform. We also note, by standard theory, see, e.g., [24], that $\Gamma_2^{\mathbb{Z}_+}$ is homeomorphic to the Cantor set X in Example 2.2 above. In particular, it is known that the compact space $\Gamma_N^{\mathbb{Z}_+}$ is totally disconnected.

Lemma 3.9. *Let $N\in\mathbb{Z}_+$, $N\geq 2$, and let $\mathcal{O}_N$ be the Cuntz C^*-algebra with generators $\{s_i\}_{0\leq i<N}$ subject to the axioms in Definition* 3.6*, i.e.,* (3.8)*. The C^*-algebra $\mathcal{C}$ is the norm-closure of the algebra generated by the elements*

$$e(a):=s_as_a^*=s_{a_1}s_{a_2}\cdots s_{a_k}s_{a_k}^*\cdots s_{a_2}^*s_{a_1}^*, \tag{3.26}$$

where $k\in\mathbb{Z}_+$ and $a=(a_1,\dots,a_k)\in\Gamma_N^k$. Let $\xi_i\colon X\to\Gamma_N$ be the coordinate function (3.22)*, $X=\Gamma_N^{\mathbb{Z}_+}$. Then the assignment*

$$\left(\chi_{\{a_1\}}\circ\xi_1\right)\left(\chi_{\{a_2\}}\circ\xi_2\right)\cdots\left(\chi_{\{a_k\}}\circ\xi_k\right)\xrightarrow{\;G\;}e(a_1,\dots,a_k) \tag{3.27}$$

extends to a C^*-isomorphism of $C(X)$ onto $\mathcal{C}$.

PROOF. The function f_a, for $a\in\Gamma_N^k$, in the formula on the left-hand side in (3.27) is given as follows: Evaluation at $x=(x_i)\in X$, $f_a(x)=\delta_{a_1,x_1}\delta_{a_2,x_2}\cdots\delta_{a_k,x_k}$. From the definition of the Tychonoff topology it follows that each f_a is continuous, and that the family $\{f_a\}_{k\in\mathbb{Z}_+,\,a\in\Gamma_N^k}$ separates points in X.

It follows from the relations on the generators $\{s_i\}_{0\leq i<N}$ that the assignment G in (3.27) is an isomorphism from an abelian subalgebra $\mathcal{S}$ of $C(X)$ into a dense subalgebra of $\mathcal{C}$. But $\mathcal{S}$ is dense in $C(X)$ by virtue of the Stone-Weierstraß theorem, and it is immediate from this that G extends uniquely, by closure, to a $*$-isomorphism of $C(X)$ onto $\mathcal{C}$. $\square$

Definition 3.10. Let (X,d) be a compact metric space, and let $N\in\mathbb{Z}_+$, $N\geq 2$, be given. We say that an N-adic system $\{A_k(a)\}_{k\in\mathbb{Z}_+,\,a\in\Gamma_N^k}$ of partitions of X is *affiliated with* an iterated function system (IFS) on X if there is a

system $\sigma, (\sigma_i)_{0\leq i<N}$ of continuous maps such that

$$\sigma \circ \sigma_i = \mathrm{id}_X, \qquad i \in \Gamma_N, \tag{3.28}$$

and

$$\sigma_{a_1}\sigma_{a_2}\cdots\sigma_{a_k}(X) = A_k(a) \qquad \text{for all } k \in \mathbb{Z}_+ \text{ and } a \in \Gamma_N^k. \tag{3.29}$$

(Note that (3.28) is part of the definition of an IFS.)

The following is a corollary to the result stated as Lemma 3.5 above, i.e., the construction of a projection-valued measure $E^A(\,\cdot\,)$ from a given representation (S_i) of $\mathcal{O}_N$ on a Hilbert space and a given N-adic system $(A_k(a))$ of partitions.

Corollary 3.11. *Let (S_i) be a representation of $\mathcal{O}_N$ on a Hilbert space $\mathcal{H}$, and let $(A_k(a))$ be an N-adic system of partitions which is affiliated with a continuous iterated function system σ, σ_i acting on a compact metric space (X, d). Then the projection-valued measure $E^A(\,\cdot\,)$, see Lemma 3.5, satisfies*

$$S_a E^A(B) S_a^* = E^A(\sigma_a B) \qquad \text{for all } B \in \mathcal{B}(X), \quad k \in \mathbb{Z}_+, \text{ and all } a \in \Gamma_N^k, \tag{3.30}$$

and

$$\sum_{i=0}^{N-1} S_i E^A(B) S_i^* = E^A\left(\sigma^{-1}(B)\right), \tag{3.31}$$

where $a = (a_1, \dots, a_k)$, $\sigma_a = \sigma_{a_1} \circ \cdots \circ \sigma_{a_k}$, and

$$\sigma^{-1}(B) = \{\, x \in X \mid \sigma(x) \in B \,\}.$$

PROOF. The argument in the proof of (3.30) and (3.31) is based directly on the two-step approximation which went into the construction of the measure $E^A(\,\cdot\,)$; see Lemma 3.5 for details.

With the assumptions on the representation (S_i) and the IFS partition, the two operator commutation relations (3.30)–(3.31) follow from the same approximation, coupled with the observation that if $a \in \Gamma_N^k$ and $b \in \Gamma_N^l$, then

$$\begin{aligned} S_a E^A(A_l(b)) S_a^* &= S_a S_b S_b^* S_a^* = S_{ab} S_{ab}^* \\ &= E^A(A_{k+l}(ab)) = E^A(\sigma_a(A_l(b))), \end{aligned} \tag{3.32}$$

where $ab = (a_1, \dots, a_k, b_1, \dots, b_l) \in \Gamma_N^{k+1}$, i.e., concatenation, and the formula

$$\bigcup_i \sigma_i B = \sigma^{-1}(B), \qquad B \in \mathcal{B}(X). \tag{3.33}$$

□

4 Endomorphisms of $B(\mathcal{H})$

From the point of view of the pure mathematics of operator algebras, it is natural to ask about the von Neumann type of the representations of the Cuntz algebras that come from subband filters (i.e., are they type I, II, or III, and how do they decompose?) While this is addressed in [27], and to some extent (in a different context) in [23], [25], and [26], we will not discuss it here. Rather we will address a related question regarding the selection of the "best" wavelets in specific parametrized families.

Let $N \in \mathbb{Z}_+$, $N \geq 2$, and let $\mathcal{H}$ be a complex Hilbert space. In our understanding of scaling problems in approximation theory, it is often helpful to study endomorphisms of the C^*-algebra of all bounded operators on $\mathcal{H}$. By this we mean a linear mapping $\alpha \colon B(\mathcal{H}) \to B(\mathcal{H})$ taking $\mathbb{1}_{\mathcal{H}}$ to $\mathbb{1}_{\mathcal{H}}$ and satisfying

$$\alpha(ST) = \alpha(S)\,\alpha(T), \; S, T \in B(\mathcal{H}), \quad \text{and} \quad \alpha(T^*) = \alpha(T)^*. \tag{4.1}$$

We showed in [25] that there is a correspondence between $\operatorname{End}(B(\mathcal{H}))$ and representations of the Cuntz relations; see (3.8) in Definition 3.6. If $(S_i)_{0 \leq i < N}$ is a representation of $\mathcal{O}_N$ on a Hilbert space $\mathcal{H}$, then define $\alpha \colon B(\mathcal{H}) \to B(\mathcal{H})$ by

$$\alpha(T) = \sum_{i=0}^{N-1} S_i T S_i^*, \qquad T \in B(\mathcal{H}), \tag{4.2}$$

and it is clear that $\alpha \in \operatorname{End}(B(\mathcal{H}))$. Moreover, the relative commutant

$$B(\mathcal{H}) \cap \alpha(B(\mathcal{H}))' = \{\, T \in B(\mathcal{H}) \mid T\alpha(X) = \alpha(X)\,T, \; X \in B(\mathcal{H}) \,\}$$

is naturally isomorphic to the algebra of all N-by-N complex matrices $M_N(\mathbb{C})$. If $(e_{i,j})$ are the usual matrix units in $M_N(\mathbb{C})$, i.e.,

$$e_{i,j}(k,l) = \delta_{i,k}\delta_{j,l}, \tag{4.3}$$

then the assignment

$$M_N(\mathbb{C}) \ni e_{i,j} \longmapsto S_i S_j^* \in \alpha(B(\mathcal{H}))' \tag{4.4}$$

defines the isomorphism. There is a similar result for the commutant of the iterated mapping $\alpha^k(B(\mathcal{H}))$. Then there is a similar isomorphism:

$$e_{i_1,j_1} \otimes e_{i_2,j_2} \otimes \cdots \otimes e_{i_k,j_k} \longmapsto S_{i_1} S_{i_2} \cdots S_{i_k} S_{j_k}^* \cdots S_{j_2}^* S_{j_1}^*. \tag{4.5}$$

The correspondence between $\operatorname{End}(B(\mathcal{H}))$ and representations from (4.2) is not quite unique: If the (S_i) system is given and if $u = (u_{i,j})$ is a unitary

N-by-N matrix, then the system $S_i^u := \sum_j u_{i,j} S_j$ defines the same endomorphism $\alpha(T) = \sum_i S_i^u T S_i^{u\,*}$, but this is the extent of the non-uniqueness in the correspondence.

The following result shows that the case when the induced measure

$$\mu_f(\,\cdot\,) = \|E(\,\cdot\,)f\|^2 \tag{4.6}$$

is a product measure on

$$X_N = \Gamma_N^{\mathbb{Z}_+} \tag{4.7}$$

is exceptional. Here μ_f is the measure defined in (1.17), and $E(\,\cdot\,)$ is the projection-valued measure defined on the Borel sets in X_N which is induced from some given representation $(S_i)_{0\leq i<N}$ of $\mathcal{O}_N$. The result shows that μ_f is a product measure precisely when the vector $f \in \mathcal{H}$, $\|f\| = 1$, is a simultaneous eigenvector for the operators S_i^*. The operators S_i^* have the form $\textcircled{\downarrow}$[filter], the two operator on the left in Figure 1.1, which is the case $N = 2$, i.e., filter followed by down-sampling; see Figure 1.1 in Section 1.

Motivated by the main result in [27], it is appropriate to restrict attention to irreducible representations when considering the representations of the Cuntz algebras induced by subband filters from signal processing: this is needed in fact for the implication (i)⇒(ii) in the proposition below. It is needed again for the states ω_f from (4.11); see also [23].

Proposition 4.1. (*see* [25]) *Let $(S_i)_{0\leq i<N}$ be a representation of $\mathcal{O}_N$ on a Hilbert space $\mathcal{H}$, and let $\alpha = \alpha_S$ be the corresponding endomorphism of $B(\mathcal{H})$ (see (4.2)). Let $f \in \mathcal{H}$, $\|f\| = 1$, and let $\omega_f(\,\cdot\,) = \langle\, f \mid \cdot\, f\rangle$ be the corresponding state. The following three conditions are equivalent.*

(i) *$\omega_f(\alpha(T)) = \omega_f(T)$ for all $T \in B(\mathcal{H})$.*
(ii) *f is a joint eigenvector for S_i^*, $0 \leq i < N$.*
(iii) *There are $\lambda_i \in \mathbb{C}$, $0 \leq i < N$, with $\sum_{i=0}^N |\lambda_i|^2 = 1$, such that*

$$\omega_f\left(S_{i_1}S_{i_2}\cdots S_{i_k}S_{j_l}^*\cdots S_{j_2}^*S_{j_1}^*\right) = \bar{\lambda}_{i_1}\bar{\lambda}_{i_2}\cdots\bar{\lambda}_{i_k}\lambda_{j_1}\lambda_{j_2}\cdots\lambda_{j_l} \tag{4.8}$$

for all $k, l \in \mathbb{Z}_+$ and all $i_i, \dots, i_k \in \Gamma_N$ and $j_1, \dots, j_l \in \Gamma_N$.

If the representation is assumed to be type I*, then a fourth condition is equivalent to the first three:*

(iv) *The measure μ_f obtained from ω_f by restriction to the maximal abelian subalgebra $\mathcal{C}$ is a product measure on $X_N = \Gamma_N^{\mathbb{Z}_+}$.*

In general (iii)⇒(iv)*.*

Remark 4.2. If (i)–(iii) hold, the λ_i in (iii) are the eigenvalues from (ii).

Remark 4.3. For non-type-I representations, it is possible to have (iv) satisfied, even if (i)–(iii) fail to hold.

PROOF. The equivalence of conditions (i), (ii), and (iii) was already established in [25]. Indeed, if f in $\mathcal{H}$ satisfies (ii), there are $\lambda_i \in \mathbb{C}$ such that

$$S_i^* f = \lambda_i f. \tag{4.9}$$

Using

$$\omega_f \left(S_{i_1} \cdots S_{i_k} S_{j_l}^* \cdots S_{j_1}^* \right) = \left\langle \, S_{i_k}^* \cdots S_{i_1}^* f \;\middle|\; S_{j_l}^* \cdots S_{j_1}^* f \right\rangle ,$$

formula (4.8) in (iii) follows. The proof that (i)⇒(ii) relies on the fact that $\omega_f(\,\cdot\,)$ is a pure state on $B(\mathcal{H})$; see [25]. Now if the relations (4.9) are substituted into

$$\sum_i S_i S_i^* = \mathbb{1}_{\mathcal{H}} \,, \tag{4.10}$$

we get $\sum_i |\lambda_i|^2 = 1$. Setting $p_i = |\lambda_i|^2$, we get a probability distribution on Γ_N. Setting $k = l$ and $i_1 = j_1, \ldots, i_k = j_k$ in (4.8), we finally conclude that

$$\omega_f \left(S_{i_1} \cdots S_{i_k} S_{i_k}^* \cdots S_{i_1}^* \right) = p_{i_1} p_{i_2} \cdots p_{i_k} \,. \tag{4.11}$$

This implies that $\mu_f = \omega_f|_{\mathcal{C}}$ is a product measure. Indeed, setting $\mu_f = \omega_f|_{\mathcal{C}}$, and using Lemma 3.9, we get

$$\mu_f \left(\{\, x \in X_N \mid x_1 = i_1, \ldots, x_k = i_k \,\} \right) = p_{i_1} p_{i_2} \cdots p_{i_k} \,, \tag{4.12}$$

which is to say that μ_f is the product measure determined by the probability distribution $(p_i)_{0 \leq i < N}$. Equivalently, μ_f is the unique probability measure on X_N which satisfies the identity (3.24).

The conclusion in (iv) is the fact that $\mu_f = \omega_f|_{\mathcal{C}}$ is a product measure. Suppose the probabilities are $(p_i)_{i \in \Gamma_N}$. Then (4.12) holds. But in terms of the $\mathcal{O}_N$ representation, this reads as (4.11).

Now suppose the representation is type I. Using again that ω_f is pure, we conclude that (4.9) must hold for some $\lambda_i \in \mathbb{C}$ with $|\lambda_i| = p_i$.

To show that there are non-type-I representations of $\mathcal{O}_N$ for which (iv) holds, but (iii) does not, it is enough to display a state ω on $\mathcal{O}_N$ for which (4.11) holds, but (4.8) fails. Such states are known and studied in [23]. They are called KMS states. We show that for every $p_i > 0$, such that $\sum_i p_i = 1$, there is a state $\omega = \omega_{(p)}$ on $\mathcal{O}_N$ for which $\omega \left(S_{i_1} \cdots S_{i_k} S_{j_l}^* \cdots S_{j_1}^* \right) = \delta_{k,l} \delta_{i_1,j_1} \cdots \delta_{i_k,j_k} p_{i_1} \cdots p_{i_k}$. Note that this is consistent with (4.11) so $\omega|_{\mathcal{C}}$ is a product measure on $X_N = \Gamma_N^{\mathbb{Z}_+}$, but it is inconsistent with (4.8). It is known that the representation generated by ω is type III, i.e., it generates a type III von Neumann algebra. □

Example 4.4. We now calculate the measure μ_f from the dyadic partitions of the unit interval $[0,1)$ of Example 2.1 in the case of two specific representations of $\mathcal{O}_2$. Each of the representations yields a product measure on the Cartesian product space $X_2 = \Gamma_2^{\mathbb{Z}_+}$: the first (a) has probability weights $p_0 = 1$, $p_1 = 0$, and the second (b) has $p_0 = p_1 = 1/2$. The induced measure on $[0,1)$ for (a) is the Dirac mass δ_0 on $[0,1)$, i.e., for the first representation; and it is the restricted Lebesgue measure in the second case (b). We take $\mathcal{H} = L^2(\mathbb{T})$ as the Hilbert space for both representations. We introduce the notation

$$e_k(z) := z^k \tag{4.13}$$

for the Fourier basis on $\mathcal{H}$, i.e., $\{e_k\}_{k\in\mathbb{Z}}$ is the usual orthonormal basis of $L^2(\mathbb{T})$ from Fourier analysis. Following the discussion of Section 1, we set

$$S_i f(z) = m_i(z) f\left(z^2\right), \qquad f \in L^2(\mathbb{T}),\ z \in \mathbb{T},\ i = 0, 1, \tag{4.14}$$

where m_0, m_1 is a system of quadrature mirror filters. The conditions on the two functions may be summarized in the requirement that

$$\text{the matrix} \quad \frac{1}{\sqrt{2}} \begin{pmatrix} m_0(z) & m_0(-z) \\ m_1(z) & m_1(-z) \end{pmatrix} \quad \text{is unitary for a.a. } z \in \mathbb{T}. \tag{4.15}$$

The two cases are (a) and (b) below.

(a) **A permutative representation** (see [26]).

$$\text{With} \quad \left\{ \begin{matrix} m_0 = e_0, \\ m_1 = e_1, \end{matrix} \right\} \quad \text{we have} \quad \left\{ \begin{matrix} S_0^* e_0 = e_0, \\ S_1^* e_0 = 0. \end{matrix} \right.$$

(b) **The representation of the Haar wavelet** (see [7]).

$$\text{With} \quad \left\{ \begin{aligned} m_0 &= \frac{1}{\sqrt{2}}(e_0 + e_1), \\ m_1 &= \frac{1}{\sqrt{2}}(e_0 - e_1), \end{aligned} \right\} \quad \text{we have} \quad \left\{ \begin{aligned} S_0^* e_0 &= \frac{1}{\sqrt{2}} e_0, \\ S_1^* e_0 &= \frac{1}{\sqrt{2}} e_0. \end{aligned} \right.$$

Hence in case (a), the measure μ_{e_0} is the Dirac mass at $x = 0$ in $[0,1)$, or $\mu_{e_0} = \delta_0$, and in case (b), the measure μ_{e_0} is Lebesgue measure on $[0,1)$.

The representation of $\mathcal{O}_2$ described in (a) above is *permutative* in the sense of [26]. A permutative representation $(S_i)_{i=0,1}$ of $\mathcal{O}_2$ in a Hilbert space $\mathcal{H}$ is one for which $\mathcal{H}$ has an orthonormal basis $\{e_n\}_{n\in\mathbb{Z}}$ such that each of the isometries S_i maps the basis to itself, i.e., there are maps $\sigma_i \colon \mathbb{Z} \to \mathbb{Z}$ such that

$$S_i e_n = e_{\sigma_i(n)}, \qquad i \in \{0, 1\},\ n \in \mathbb{Z}. \tag{4.16}$$

For the representation given in (a), the two maps σ_i are $\sigma_0 n = 2n$, $\sigma_1 n = 2n + 1$. For permutative representations, the problem of diagonalizing the commutative family of operators

$$S_{i_1} S_{i_2} \cdots S_{i_k} S_{i_k}^* \cdots S_{i_2}^* S_{i_1}^* \tag{4.17}$$

is very simple; see [26]. But unfortunately wavelet representations are typically not permutative, and the reader is referred to [27] for details of the argument.

In general, the explicit transform which diagonalizes the commuting family (4.17) may be somewhat complicated. But if the representation (S_i) is permutative, it is easy to see that the operator monomials from (4.17) may be naturally realized as multiplication operators on the sequence space $\ell^2(\mathbb{Z})$. Specifically, if $k \in \mathbb{Z}_+$, and $(i_1, \dots, i_k) \in \Gamma_2^k$ are given, then the corresponding operator in (4.17) is represented as multiplication by the indicator function of the set $\sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_k}(\mathbb{Z}) \subset \mathbb{Z}$, where the maps σ_i are determined from the formula (4.16).

5 Computation of μ_f

We now turn to the computation of the measures $\mu_f(\,\cdot\,) = \|E(\,\cdot\,) f\|^2$ in the special case when the representation of $\mathcal{O}_N$ arises from a system of subband filters. Recall from Section 3 that every representation of $\mathcal{O}_N$ defines a projection-valued measure on $[0,1)$ when restricted to the subalgebra $\mathcal{C}$ in $\mathcal{O}_N$. A system of subband filters corresponding to N subbands is a set of L^∞-functions $m_0, m_1, \dots, m_{N-1}$ on $\mathbb{T}$ such that the following matrix function on $\mathbb{T}$ takes unitary values:

$$\frac{1}{\sqrt{N}} \left(m_j \left(z e^{i2\pi \frac{k}{n}} \right) \right)_{0 \le j,k < N} . \tag{5.1}$$

Specifically, for a.e. $z \in \mathbb{T}$, the $N \times N$ matrix of (5.1) is assumed unitary.

The following lemma is well known; see [9].

Lemma 5.1. *Let $m_0, m_1, \dots, m_{N-1}$ be in $L^\infty(\mathbb{T})$ and set*

$$S_j f(z) = m_j(z) f\left(z^N\right), \quad f \in L^2(\mathbb{T}),\ z \in \mathbb{T},\ j = 0, 1, \dots, N-1. \tag{5.2}$$

Then $(S_j)_{0 \le j < N}$ is a representation of $\mathcal{O}_N$ on the Hilbert space $L^2(\mathbb{T})$ if and only if the functions m_j satisfy the unitarity property (5.1).

We state the next result for the middle-third Cantor set, but it applies *mutatis mutandis* to most of the fractals based on iterated function systems (IFS's)

built on affine maps.

Proposition 5.2. (Example 2.2 revisited.)

Let

$$\begin{cases} m_0 = \dfrac{1}{\sqrt{2}}\left(e_0 + e_2\right), \\ m_1 = e_1, \\ m_2 = \dfrac{1}{\sqrt{2}}\left(e_0 - e_2\right), \end{cases} \tag{5.3}$$

and let

$$S_j f\left(z\right) = m_j\left(z\right) f\left(z^3\right), \qquad j = 0, 1, 2, \ f \in L^2\left(\mathbb{T}\right), \ z \in \mathbb{T}, \tag{5.4}$$

where $e_p\left(z\right) = z^p$, $p \in \mathbb{Z}$, and $L^2\left(\mathbb{T}\right)$ is the Hilbert space $\mathcal{H}$ of L^2-functions on $\mathbb{T}$ defined from the Haar measure on $\mathbb{T}$. Then $\left(S_j\right)_{j=0,1,2}$ is a representation of $\mathcal{O}_3$ on $\mathcal{H} = L^2\left(\mathbb{T}\right)$. Let I be the unit interval, and let $E\left(\,\cdot\,\right)$ be the corresponding projection-valued measure on $\mathcal{B}\left(I\right)$. Then the induced scalar measure $\mu_{e_0}\left(\,\cdot\,\right) = \left\|E\left(\,\cdot\,\right)e_0\right\|^2$ is the middle-third Cantor measure of Example 2.2*, i.e., the unique measure μ on I which satisfies* (2.9)*.* (*It is supported on the middle-third Cantor set X.*)

PROOF. That the system $\left(S_j\right)_{j=0,1,2}$ of (5.4) forms a representation of $\mathcal{O}_3$ on $\mathcal{H}$ follows immediately from Lemma 5.1. As noted in Lemma 3.7, the corresponding projection-valued measure E is determined as follows: If $k \in \mathbb{Z}_+$, and $a = \left(a_1, a_2, \dots, a_k\right) \in \Gamma_3^k$, then recall that

$$E\left(J_k\left(a\right)\right) = S_a S_a^*, \tag{5.5}$$

where $J_k\left(a\right) = \left[\dfrac{a_1}{3} + \dots + \dfrac{a_k}{3^k}, \dfrac{a_1}{3} + \dots + \dfrac{a_k}{3^k} + \dfrac{1}{3^k}\right)$, and $S_a := S_{a_1} S_{a_2} \cdots S_{a_k}$.
From (5.3), we get $\left\{\begin{array}{l} S_0^* e_0 = \dfrac{1}{\sqrt{2}} e_0, \\ S_1^* e_0 = 0, \\ S_2^* e_0 = \dfrac{1}{\sqrt{2}} e_0, \end{array}\right\}$, which are the joint eigenvalue identities of (ii) in Proposition 4.1. Now a direct check on $\mu_{e_0}\left(\,\cdot\,\right) = \left\|E\left(\,\cdot\,\right)e_0\right\|^2$, using (5.5) and Proposition 4.1, (ii)$\Rightarrow$(iv), shows that μ_{e_0} is indeed the Cantor measure of Example 2.2. See also [23]. $\square$

For more about the representation (5.4) and the corresponding fractal wavelet, the reader is referred to [28]. While this representation does not correspond to a system of wavelet functions φ, ψ_1, ψ_2 in $L^2\left(\mathbb{R}\right)$, we show in [28] that

there is a Hilbert space of functions on $\mathbb{R}$ which admits φ, ψ_1, ψ_2 as wavelet generators. If $s = \log_2(3) = \ln 2/\ln 3$, the wavelet system is constructed on the Hausdorff measure $\mathcal{H}_s$, i.e., the measure on $\mathbb{R}$ constructed from $(dx)^s$ by the usual completion; see also [29] for details on the Hausdorff measure $\mathcal{H}^s$.

Some terminology: For functions g on $\mathbb{T}$, we define the Fourier transform $\hat{g}(n)$ as follows:

$$\hat{g}(n) = \langle\, e_n \mid g \,\rangle = \int_{\mathbb{T}} z^{-n} g(z)\, d\lambda(z) = \int_0^1 e^{-i2\pi n\theta} g(\theta)\, d\theta, \tag{5.6}$$

where λ denotes Haar measure on $\mathbb{T}$, and where we have identified $g(\theta)$ with $g\left(e^{i2\pi\theta}\right)$.

If $k \in \mathbb{Z}_+$, and $a = (a_1, \dots, a_k) \in \Gamma_N^k$, set

$$m_a(z) := m_{a_1}(z)\, m_{a_2}\left(z^N\right) \cdots m_{a_k}\left(z^{N^{k-1}}\right), \tag{5.7}$$

or in additive notation,

$$m_a(\theta) := m_{a_1}(\theta)\, m_{a_2}(N\theta) \cdots m_{a_k}\left(N^{k-1}\theta\right). \tag{5.8}$$

When a system m_j satisfies the condition (5.1) we say that the representation (5.2) is a wavelet representation of $\mathcal{O}_N$.

Proposition 5.3. *Let the functions $(m_j)_{0\le j<N}$ satisfy the condition (5.1) and let S_j be the corresponding wavelet representation of $\mathcal{O}_N$ on the Hilbert space $L^2(\mathbb{T})$. Let $f \in L^2(\mathbb{T})$, $\|f\| = 1$, and let $k \in \mathbb{Z}_+$, $a = (a_1, \dots, a_k) \in \Gamma_N^k$. Then*

$$\mu_f(J_k(a)) = \sum_{n\in\mathbb{Z}} \left|(f\bar{m}_a)\widehat{\ }\left(nN^k\right)\right|^2. \tag{5.9}$$

PROOF. Let the conditions be as stated. Then

$$\begin{aligned}
\mu_f(J_k(a)) &= \|S_a S_a^* f\|^2 = \langle\, f \mid S_a S_a^* f \,\rangle = \|S_a^* f\|^2 = \sum_{n\in\mathbb{Z}} |\langle\, e_n \mid S_a^* f \,\rangle|^2 \\
&= \sum_{n\in\mathbb{Z}} |\langle\, S_a e_n \mid f \,\rangle|^2 = \sum_{n\in\mathbb{Z}} \left|\left\langle\, m_a(z)\, e_n\left(z^{N^k}\right) \,\middle|\, f \,\right\rangle\right|^2 \\
&= \sum_{n\in\mathbb{Z}} \left|\int_{\mathbb{T}} e_{-nN^k}\, \bar{m}_a\, f\, d\lambda\right|^2 = \sum_{n\in\mathbb{Z}} \left|(\bar{m}_a f)\widehat{\ }\left(nN^k\right)\right|^2,
\end{aligned}$$

which is the desired conclusion. $\square$

Specializing to $f = e_p$, for some $p \in \mathbb{Z}$, we get for $\mu_p(\cdot) = \mu_{e_p}(\cdot) =$

$\|E(\cdot)e_p\|^2$,

$$\mu_p(J_k(a)) = \sum_{n\in\mathbb{Z}} \left|\hat{m}_a\left(p - nN^k\right)\right|^2. \tag{5.10}$$

Let $N \in \mathbb{Z}_+$, $N \geq 2$, and let $(m_j)_{0\leq j<N}$ be a subband filter system, i.e., the m_j's are functions which satisfy condition (5.1). We shall assume further that m_0 is Lipschitz of order 1 as a function on $\mathbb{T}$, and that $m_0(1) = \sqrt{N}$. In that case, there is an $L^2(\mathbb{R})$ scaling function φ such that

$$\hat{\varphi}(\xi) = \prod_{k=1}^{\infty} \frac{m_0\left(\xi/N^k\right)}{\sqrt{N}},$$

where we set $m_0(\theta) = m_0\left(e^{-i2\pi\theta}\right)$, and $\hat{\varphi}(\xi) = \int_{\mathbb{R}} e^{-i2\pi\xi x}\varphi(x)\,dx$. We will assume in addition that

$$\sum_{l\in\mathbb{Z}} |\hat{\varphi}(\xi + l)|^2 = 1. \tag{5.11}$$

It is known that (5.11) is equivalent to each of the following three conditions on φ:

(i) $\|\varphi\|^2 = \int_{\mathbb{R}} |\varphi(x)|^2\,dx = 1$,
(ii) the system $\{\,\varphi(x-k) \mid k \in \mathbb{Z}\,\}$ is orthogonal in $L^2(\mathbb{R})$, and
(iii) the operator W_φ is isometric;

see [9] for details.

Recalling the N-adic representation for $\mathbb{Z}_+$,

$$n = a_1 + a_2 N + \cdots + a_k N^{k-1}, \quad k = 1, 2, \ldots,\ a = (a_1, \ldots, a_k) \in \Gamma_N^k, \tag{5.12}$$

we get a sequence of $L^2(\mathbb{R})$-functions w_n, the wavelet packet functions of Wickerhauser, satisfying

$$\hat{w}_n(\xi) = N^{-k/2} m_{a_1}\left(\frac{\xi}{N}\right) m_{a_2}\left(\frac{\xi}{N^2}\right) \cdots m_{a_k}\left(\frac{\xi}{N^k}\right) \hat{\varphi}\left(\frac{\xi}{N^k}\right). \tag{5.13}$$

Hence, setting $\bar{n} := a_k + a_{k-1}N + \cdots + a_1 N^{k-1}$, bit-reversal, we get

$$\hat{w}_{\bar{n}}\left(N^k\xi\right) = N^{-k/2} m_a(\xi)\,\hat{\varphi}(\xi). \tag{5.14}$$

In the next lemma we shall need the following transformation T_φ^k acting on $L^2(\mathbb{R})$:

$$\left(T_\varphi^k f\right)(x) = \int_{\mathbb{R}} f(x+y)\,\overline{\varphi(N^k y)}\,dy.$$

Lemma 5.4. *Let $m_0, \ldots, m_{N-1}$ and φ be as described above. Then*

$$\hat{m}_a\left(p - jN^k\right) = N^{k/2}\left(T_\varphi^k w_{\bar{n}}\right)\left(j - \frac{p}{N^k}\right). \tag{5.15}$$

PROOF.

$$
\begin{aligned}
T_{\varphi}^{k} w_{\bar{n}}\left(j-\frac{p}{N^{k}}\right) &= \int_{\mathbb{R}} e^{i2\pi\left(j-\left(p/N^{k}\right)\right)\xi}\left(T_{\varphi}^{k} w_{\bar{n}}\right)^{\widehat{}}(\xi)\ d\xi \\
&= N^{k} \int_{\mathbb{R}} e^{i2\pi\left(jN^{k}-p\right)\xi}\left(T_{\varphi}^{k} w_{\bar{n}}\right)^{\widehat{}}\left(N^{k}\xi\right)\ d\xi \\
&= \int_{\mathbb{R}} e^{i2\pi\left(jN^{k}-p\right)\xi}\hat{w}_{\bar{n}}\left(N^{k}\xi\right)\ \overline{\hat{\varphi}(\xi)}\, d\xi \\
&= N^{-k/2}\int_{\mathbb{R}} e^{i2\pi\left(jN^{k}-p\right)\xi} m_{a}(\xi)\left|\hat{\varphi}(\xi)\right|^{2}\ d\xi \\
&= N^{-k/2}\int_{0}^{1} e^{i2\pi\left(jN^{k}-p\right)\xi} m_{a}(\xi)\sum_{l\in\mathbb{Z}}\left|\hat{\varphi}(\xi+l)\right|^{2}\ d\xi \\
&= N^{-k/2}\int_{0}^{1} e^{i2\pi\left(jN^{k}-p\right)\xi} m_{a}(\xi)\ d\xi \\
&= N^{-k/2}\hat{m}_{a}\left(p-jN^{k}\right).
\end{aligned}
$$

□

Corollary 5.5. *Let $m_0, \ldots, m_{N-1}$ and φ be as described above. Then the measure $\mu_p(\cdot) = \mu_{e_p}(\cdot) = \|E(\cdot)\, e_p\|^2$ is given by the formula*

$$
\mu_{p}\left(J_{k}(a)\right) = N^{k}\sum_{j\in\mathbb{Z}}\left|\left(T_{\varphi}^{k} w_{\bar{n}}\right)\left(j-\frac{p}{N^{k}}\right)\right|^{2} \tag{5.16}
$$

for all $p\in\mathbb{Z}$, $k\in\mathbb{Z}_+$, and $a\in\Gamma_N^k$.

PROOF. The conclusion is immediate from the two previous lemmas, and the results in Section 4. □

Some consequences of the formula (5.16):

(i) It gives a formula for the measure μ_p in terms of the wavelet packet functions (w_n) themselves. It is known that the functions

$$
\left\{N^{q/2} w_{n}\left(N^{q}x-k\right)\right\} \tag{5.17}
$$

form an orthonormal basis (ONB) for $L^2(\mathbb{R})$ when the index labels n, q, and k are carefully selected: for $(n,q)\in\mathbb{N}\times\mathbb{Z}$ we may set $I(n,q) = [N^q n, N^q(n+1))$. It is known [6] that if a subset E of $\mathbb{N}\times\mathbb{Z}$ has the property that $\{\, I(n,q) \mid (n,q)\in E \,\}$ is a partition of $[0,\infty)$ with overlap on at most a countable set, then

$$
\left\{\, N^{q/2} w_{n}\left(N^{q}x-k\right) \;\middle|\; (n,q)\in E,\ k\in\mathbb{Z} \right\} \tag{5.18}
$$

is an orthonormal basis for $L^2(\mathbb{R})$. It is of interest to know when the exceptional set with overlap might be more than countable, for example if the ONB conclusion for (5.18) might hold if it is only known that the overlap of the partition sets $\{ I(n,q) \mid (n,q) \in E \}$ is at most of Lebesgue measure zero: hence the interest in when the spectral measure is absolutely continuous with respect to the Lebesgue measure on $[0,1)$.

(ii) Formula (5.16) shows that

$$\mu_{p+N^k}(J_k(a)) = \mu_p(J_k(a))$$

and

$$\sum_{p=0}^{N^k-1} \mu_p(J_k(a)) = 1.$$

(iii) Formula (5.16) shows that the size estimate of μ_p on N-adic intervals depends on the asymptotics of the sequence $\{ w_n \mid n \in \mathbb{N} \}$ as $n \to \infty$, and there are effective estimates on $\|w_n\|_{L^\infty(\mathbb{R})}$ in the literature: see, e.g., [6], [30], and [31].

(iv) Finally, (5.16) specializes to a known formula in case $N = 2$ and w_n is the Lemarié-Meyer wavelet packet; see [12].

6 The family of measures $\{ \mu_f \mid f \in \mathcal{H} \}$

Since the standard operations that are usually applied to systems of subband filters $(m_i)_{0 \le i < N}$ depend on the functions m_i having some degree of regularity, it is not surprising that new and different geometric tools are needed for the analysis when the m_i's are only known to be measurable. In addition to the present results, the reader is referred to recent papers of R. Gundy, [32] and [33].

We saw that every representation of the C^*-algebra $\mathcal{O}_N$ on a Hilbert space $\mathcal{H}$ naturally induces a family of measures $\{ \mu_f \mid f \in \mathcal{H} \}$ with each μ_f being a Borel measure on the unit interval $J = [0,1)$. We also saw that, if $(S_i)_{0 \le i < N}$ is a Haar wavelet representation of $\mathcal{O}_N$ on $\mathcal{H} = L^2(\mathbb{T})$, then the measure μ_{e_0} is the Lebesgue measure dt restricted to J. As in Section 4, we denote the Fourier basis for $L^2(\mathbb{T})$ by $e_n(z) = z^n$, $n \in \mathbb{Z}$, $z \in \mathbb{T}$.

Terminology: Let $\mathcal{C}$ be an abelian C^*-algebra of operators on a Hilbert space $\mathcal{H}$, and let $f \in \mathcal{H}$. We set $\mathcal{C}f := \{ Cf \mid C \in \mathcal{C} \}$. We denote the closure of $\mathcal{C}f$ by $[\mathcal{C}f]$, or just $\mathcal{H}_f$ when the algebra $\mathcal{C}$ is clear from the context.

A well-known fact, based on Zorn's lemma, is that there is always a family

$f_i \in \mathcal{H}$, $\|f_i\| = 1$, such that

$$\mathcal{H} = \sum_i^{\oplus} \mathcal{H}_{f_i}. \tag{6.1}$$

Implicit in (6.1) is the assertion that

$$\mathcal{H}_{f_i} \perp \mathcal{H}_{f_j} \qquad \text{when } i \neq j, \tag{6.2}$$

and that the closure of the spaces $\mathcal{H}_{f_i}$ is all of $\mathcal{H}$.

A vector f in $\mathcal{H}$ for which $\mathcal{H}_f = \mathcal{H}$ is called a *cyclic vector.*

We state the next result just for the case $N = 2$, but it holds for any $N \in \mathbb{N}$, $N \geq 2$.

Lemma 6.1. *Let* $m_0 = \frac{1}{\sqrt{2}}(e_0 + e_1)$, *and* $m_1 = \frac{1}{\sqrt{2}}(e_0 - e_1)$. *Let*

$$S_i f(z) = m_i(z) f\left(z^2\right), \qquad i = 0, 1,\ z \in \mathbb{T},\ f \in L^2(\mathbb{T}), \tag{6.3}$$

and let $\mathcal{C}$ *be the* C^**-algebra generated by the commuting projections*

$$P_a := S_a S_a^* \tag{6.4}$$

as $k \in \mathbb{N}$ *and* $a = (a_1, \dots, a_k) \in \Gamma_2^k$. *Then* e_0 *(i.e., the constant function* 1*) is a cyclic vector for* $\mathcal{C}$ *acting on the Hilbert space* $\mathcal{H} = L^2(\mathbb{T})$, *and the measures* $\{\, \mu_f \mid f \in \mathcal{H} \,\}$ *are all absolutely continuous with respect to the Lebesgue measure restricted to the unit interval* $J = [0, 1)$.

PROOF. A direct calculation, using the formula

$$S_i^* f(z) = \frac{1}{2} \sum_{\substack{w \in \mathbb{T} \\ w^2 = z}} \overline{m_i(w)}\, f(w), \tag{6.5}$$

yields

$$S_i^* e_0 = \frac{1}{\sqrt{2}} e_0, \qquad i = 0, 1, \tag{6.6}$$

$$S_i^* e_{2n} = \frac{1}{\sqrt{2}} e_n, \qquad i = 0, 1, \tag{6.7}$$

$$S_0^* e_{2n+1} = \frac{1}{\sqrt{2}} e_n,\ S_1^* e_{2n+1} = -\frac{1}{\sqrt{2}} e_n. \tag{6.8}$$

More generally, if

$$n = i_1 + 2i_2 + \cdots + 2^{k-1} i_k + 2^k p, \qquad (i_1, \dots, i_k) \in \Gamma_2^k,\ p \in \mathbb{Z}, \tag{6.9}$$

then

$$S_a^* e_n = \pm 2^{-k/2} e_p \tag{6.10}$$

for $a = (a_1, \dots, a_k) \in \Gamma_2^k$ and $S_a^* := S_{a_k}^* \cdots S_{a_1}^*$. Introducing the familiar functions

$$m_a(z) := m_{a_1}(z)\, m_{a_2}\left(z^2\right) \cdots m_{a_k}\left(z^{2^{k-1}}\right) \tag{6.11}$$

of (5.7), we see that

$$S_a S_a^* e_0 = \pm 2^{-k/2} m_a. \tag{6.12}$$

Let $f = \sum_{n\in\mathbb{Z}} \xi_n e_n \in L^2(\mathbb{T})$, and suppose $\langle\, f \mid S_a S_a^* e_0 \,\rangle = 0$ for all $k \in \mathbb{N}$ and all $a \in \Gamma_2^k$. Then

$$\langle\, S_a^* f \mid e_0 \,\rangle = 0 \text{ for all } a; \quad \text{or equivalently } \int_{\mathbb{T}} S_a^* f \, d\lambda = 0 \text{ for all } a. \tag{6.13}$$

But $S_a^* f = \sum_{n\in\mathbb{Z}} \xi_n S_a^* e_n$, and using (6.13) and (6.10), we conclude that $\xi_n = 0$ for all $n \in \mathbb{Z}$, and therefore $f = 0$. This means that the closed span of the vectors

$$\left\{\, S_a S_a^* e_0 \mid k \in \mathbb{N},\ a \in \Gamma_2^k \,\right\} \tag{6.14}$$

is all of $L^2(\mathbb{T})$. Hence, for every $h \in L^2(\mathbb{T})$, the space $[\,\mathcal{C}h\,]$ is contained in $[\,\mathcal{C}e_0\,] = L^2(\mathbb{T})$; and the absolute continuity of μ_h follows from this, since we know that μ_{e_0} is the Lebesgue measure on the unit interval. $\square$

When the lemma is combined with the next theorem, we get the following result for the Haar wavelet representation.

Proposition 6.2. *Let $(S_i)_{i=0,1}$ be the Haar wavelet representation of $\mathcal{O}_2$ acting on $L^2(\mathbb{T})$. Then there is a unique unitary isometry*

$$V \colon L^2\left([0,1)\,,\, dt\right) \longrightarrow L^2(\mathbb{T}) \tag{6.15}$$

such that

$$V\left(\chi_{J_k(a)}\right) = S_a S_a^* e_0 \tag{6.16}$$

for all $k \in \mathbb{N}$, $a = (a_1, \dots, a_k) \in \Gamma_2^k$, where $\chi_{J_k(a)}$ is the indicator function of

$$J_k(a) = \left[\frac{a_1}{2} + \cdots + \frac{a_k}{2^k}, \frac{a_1}{2} + \cdots + \frac{a_k}{2^k} + \frac{1}{2^k}\right). \tag{6.17}$$

In particular, the isometry V of (6.15) *maps* onto *the Hilbert space $L^2(\mathbb{T})$, and*

$$V^* S_a S_a^* V = M_{\chi_{J_k(a)}}, \tag{6.18}$$

where the operator on the right-hand side in (6.18) *is multiplication by the function $\chi_{J_k(a)}$ acting on $L^2\left([0,1)\,,\, dt\right)$.*

Theorem 6.3. *Let $N \in \mathbb{N}$, and let $\{A_k(a)\}_{k\in\mathbb{N},\, a\in\Gamma_N^k}$ be an N-adic system of partitions of a compact metric space X. Let $(S_i)_{0\le i<N}$ be a representation of $\mathcal{O}_N$ on a Hilbert space $\mathcal{H}$, and let $E^A(\,\cdot\,)$ be the corresponding projection-valued measure, as given by Lemma 3.5.*

(a) *Then there is a set $f_1, f_2, \dots$ (possibly finite), $f_i \in \mathcal{H}$, $\|f_i\| = 1$, such that the measures*
$$\mu_i(\,\cdot\,) := \left\| E^A(\,\cdot\,) f_i \right\|^2 \tag{6.19}$$
are mutually singular.

(b) *For each i, there is a unique isometry*
$$V_i\colon L^2(X, \mu_i) \longrightarrow \mathcal{H} \tag{6.20}$$
satisfying the following three conditions:
$$V_i \chi_{A_k(a)} = S_a S_a^* f_i \qquad \text{for } k \in \mathbb{N},\ a \in \Gamma_N^k, \tag{6.21}$$
$$V_i^* S_a S_a^* V_i = M_{\chi_{A_k(a)}}, \tag{6.22}$$
and
$$V_i\left(L^2(X,\mu_i)\right) = \mathcal{H}_{f_i}. \tag{6.23}$$

(c) *Moreover, $\mathcal{H} = \sum_i^{\oplus} \mathcal{H}_{f_i}$, where $\mathcal{H}_{f_i} := [\mathcal{C} f_i]$.*

PROOF. The vectors f_i may be chosen such that (c) holds by an application of Zorn's lemma. With this choice, it follows from [22] that the corresponding measures μ_i in (6.19) will be mutually singular.

When k is fixed, the projections $P_k(a) = S_a S_a^*$ are mutually orthogonal, with the multi-index a ranging over Γ_N^k, and $E^A(A_k(a)) = P_k(a)$. Now consider the functions from (3.12). We calculate
$$\begin{aligned}
\int_X \left| \sum_a C_a \chi_{A_k(a)} \right|^2 d\mu_i &= \int_X \sum_a |C_a|^2 \chi_{A_k(a)}\, d\mu_i \\
&= \int_X \sum_a |C_a|^2 \mu_i(A_k(a)) \\
&= \int_X \sum_a |C_a|^2 \|P_k(a) f_i\|^2 \\
&= \left\| \sum_a C_a P_k(a) f_i \right\|^2 .
\end{aligned}$$

This proves that an isometry V_i, in (6.20), is well defined. The argument is in fact the same measure-completion process which was used in Section 3. Moreover, it follows from the construction that V_i satisfies (6.21)–(6.23). $\square$

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